The twisting boundary of the Maskit slice

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1 Introduction

The Maskit slice of quasifuchsian space can be embedded as a simply connected subset of $\mathbb{C}$ bounded by a simple closed curve. At almost every point in the boundary of such a set, the boundary is *twisting* – spiralling infinitely in both directions – or *conformal* – roughly speaking, with well defined tangents. The numerical evidence presented in this paper suggests that in the case of the Maskit slice it is twisting at almost every point in its boundary. We also suggest an approach to proving this conjecture, and present an argument that suggests the Hausdorff dimension of the Maskit slice should be less than 1.25.

Figure 1 shows a detail from the Maskit slice. Figure 2 shows a sequence of zooms into the Maskit slice.

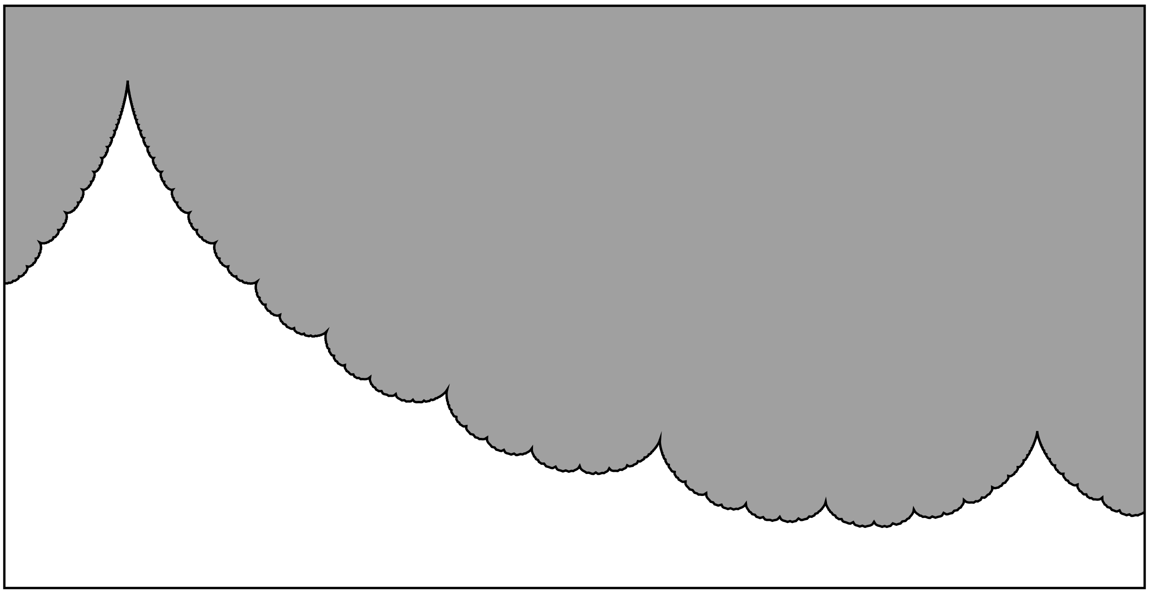

Figure 1: Detail from the Maskit slice, with the interior shaded grey

The Maskit slice is a one-complex dimensional subset of the two-complex dimensional quasifuchsian space $\mathcal{QF}$. Roughly speaking, $\mathcal{QF}$ is the space of hyperbolic geometries of a quasifuchsian manifold. A quasifuchsian manifold is, again roughly speaking, a hyperbolic 3-manifold obtained by taking a once-punctured torus crossed with an interval. In [Goodman06] we showed that there is a dense, uncountable set of points about which the boundary is spiralling infinitely. Unfortunately, the set of points about which we showed this is a countable union of Cantor sets and therefore has zero measure. Acts ergodically?

McMillan's Twist Theorem (2.4.1 below) says that at almost every point in the boundary of the Maskit slice, the boundary is twisting or conformal. The *mapping class group* of the torus acts ergodically on the boundary of the Maskit slice, which suggests intuitively that either it is twisting almost everywhere, or it is conformal almost everywhere. In section 3.3.2 we suggest a method for

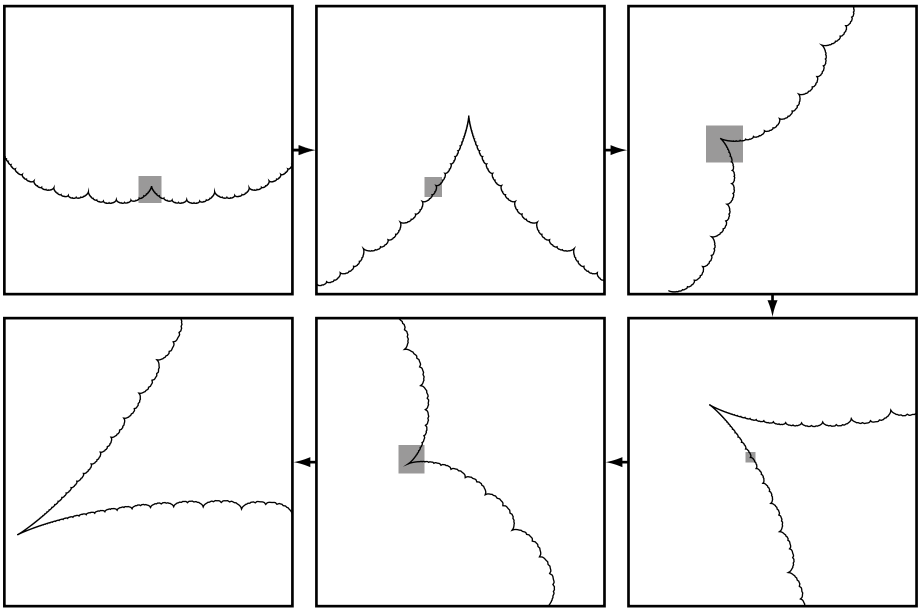

Figure 2: Successive zooms into the Maskit slice suggesting spiralling. The figures go from the top left to the bottom left in a clockwise order. The grey rectangles show the area being zoomed to in the next frame.

proving that the boundary is twisting almost everywhere by showing directly that it is not conformal almost everywhere. section refs

It seems that the key to proving this result is likely to be proving that the set of trace functions (defined below) are a set of uniform local coordinates for the Maskit slice in the sense of section 3.1 below. As you can see in figures 1 and 2, there is a special subset of what are called cusp points in the Maskit slice. These are the points at which the boundary points inwardly reaching a sharp point. McMullen showed in [McMullen91] that these points are dense in the boundary, and Miyachi showed in [Miyachi03] that the boundary is $(2, 3)$-cuspidal at these points – that is they are locally approximately like a scaled, rotated and translated copy of the curve $x^2 = y^3$. At each cusp point there is an associated trace function, which is a coordinate function in a neighbourhood of the cusp point. If these trace functions are coordinates in a sufficiently large region around each cusp point, and they behave nicely in a uniform way (as defined in section 3.1) then we can use known results about the behaviour of the boundary near cusp points to prove results about other points in the boundary by considering them as the limit of a sequence of cusp points. In fact, each cusp point is associated to a fraction p/q (giving the correct order of cusp points in the boundary), and any other point in the boundary can be associated to a real number ω with the sequence of its continued fraction convergents $p_n/q_n \to \omega$. It turns out that the behaviour of the boundary at a point corresponding to ω is governed by the partial quotients of ω – the integers in the continued fraction expansion of ω – and the behaviour of the derivatives of the trace functions at their cusp points (the local scaling factor of the local coordinates).

In section 2 of this paper we give some background definitions and results about quasifuchsian space, the Maskit slice, McMillan's twisting theorem, continued fractions and Diophantine approximation. In section 3 we present the conjectures and the numerical evidence in support of them.

2 Background

2.1 Quasifuchsian space

2.1.1 Kleinian groups and quasifuchsian manifolds

A Kleinian group is a discrete subgroup $G \leq \mathrm{PSL}_2\mathbb{C}$. The *regular set*, or *domain of discontinuity* of G is the largest set $\Omega \subseteq \hat{\mathbb{C}}$ on which G acts properly discontinuously. The complement $\Lambda = \hat{\mathbb{C}} - \Omega$ is called the *limit set* of G. If G has an invariant disc $\Delta \subseteq \hat{\mathbb{C}}$ then G is a *Fuchsian group* and Λ is contained in the boundary of the disc.

For finitely generated G, the quotient manifold Ω/G is a finite union of Riemann surfaces of finite type (see [Ahlfors64] and [Kapovich01]). If Ω/G consists of two once-punctured tori then G is called a *quasifuchsian once-punctured torus group*. In this case, Ω consists of two connected, simply connected, G-invariant components Ω^+ and Ω^- such that each of $\Omega^\pm/G$ is a once punctured torus. The limit set will be a topological circle separating $\Omega^\pm$. The group G will be a free group on two generators $\langle A, B\rangle$, where the commutator $[A, B]$ is parabolic, which we can identify with the fundamental group of a once punctured torus. Write Σ for a fixed once-punctured torus. The group G acts on the hyperbolic upper half space $\mathbb{H}^3$, and we define the manifold

$$M = (\mathbb{H}^3 \cup \Omega)/G.$$

This is a *quasifuchsian manifold* and is homeomorphic to $\Sigma \times [0, 1]$. The boundary has two components $\Omega^\pm/G$ corresponding to $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$. The leftmost manifold, M_1, in figure 3 gives a schematic view of this. See [MatTan98], [Marden74] and [Marden06] for details.

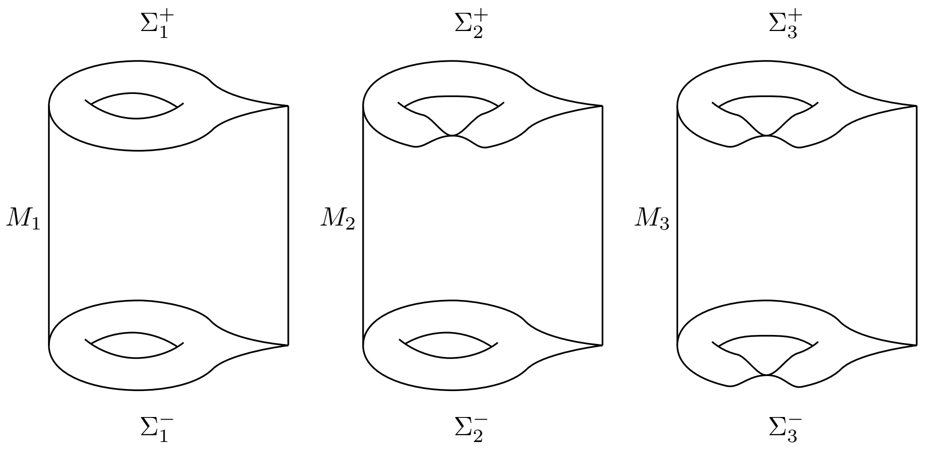

Figure 3: Various manifolds

We can consider the group G along with a choice of generators A, B as a discrete representation

$$\rho : \pi_1(\Sigma) = \langle X, Y\rangle \longrightarrow \mathrm{PSL}_2\mathbb{C},$$

such that $\rho([X, Y])$ is parabolic. Here we can think of X and Y as abstract symbols, or as generators of $\pi_1(\Sigma)$.

2.1.2 Deformation spaces

Thinking of once-punctured torus groups as representations subject to certain constraints gives us a nice way of defining the space of once-punctured torus groups. This *deformation space* is defined as follows. See [Kapovich01], [MatTan98] and [Marden06] for details. First of all, let

$$\Gamma = \pi_1(\Sigma) = \langle X, Y\rangle.$$

Now define

$$\mathcal{R}(\Gamma) = \mathrm{Hom}(\Gamma, \mathrm{PSL}_2C)/\mathrm{PSL}_2C$$

to be the representation space of Γ modulo conjugation by $\mathrm{PSL}_2\mathbb{C}$. Similarly, we define

$$\mathcal{R}_{\mathrm{p}}(\Gamma) = \{[\rho] \in \mathcal{R}(\Gamma) : \rho([X,Y]) \text{ is parabolic}\}.$$

We define *quasifuchsian space* $\mathcal{QF}$ to be the set of those classes of representations $[\rho] \in \mathcal{R}_{\mathrm{p}}(\Gamma)$ whose images are quasifuchsian once-punctured torus groups.

The Teichmüller space $\mathrm{Teich}(\Sigma)$ of Σ is the space of marked complex structures on Σ with the Teichmüller metric. A marking is just an ordered choice of generators of $\pi_1(\Sigma)$. The Teichmüller metric is defined in terms of quasiconformal maps, but we need only note here that $\mathrm{Teich}(\Sigma)$ is isometric to the hyperbolic upper half plane $\mathbb{H} = \mathbb{H}^2$. See [ImaTan92], [Lehto87] and [Bers70] for details.

Bers' Simultaneous Uniformisation theorem [Bers60] implies that $\mathcal{QF}$ is conformally equivalent to $\mathrm{Teich}(\Sigma) \times \mathrm{Teich}(\overline{\Sigma})$ (where $\overline{\Sigma}$ is Σ with the reverse orientation). Elements of $\mathcal{QF}$ are characterised by the Teichmüller parameters $\nu^{\pm}$ of the two boundary components $\Omega^{\pm}/G$.

We can complete $\mathcal{QF}$ to $\overline{\mathcal{QF}}$. We define $\overline{\mathcal{QF}}$ to be the algebraic closure of $\mathcal{QF}$ in $\mathcal{R}_{\mathrm{p}}(\Gamma)$. In a neighbourhood of $\overline{\mathcal{QF}}$, $\mathcal{R}_{\mathrm{p}}(\Gamma)$ is a smooth complex variety of dimension 2 (see [Kapovich01]). In figure 3, the manifolds M_2 and M_3 are elements in the boundary of $\overline{\mathcal{QF}}$.

originally had a ref here to section on end invariants. put something here?

Although $\mathcal{QF}$ is conformally equivalent to $\mathbb{H} \times \mathbb{H}$, the completion $\overline{\mathcal{QF}}$ has an extremely complicated structure. In particular, the boundary is not a simple curve. There are now many papers detailing the complicated way in which $\overline{\mathcal{QF}}$ self-intersects, see for example [BromHolt01].

2.1.3 End invariants

If $[\rho] \in \overline{\mathcal{QF}}$ then any component of Ω can be reached either by going to the $+$ end or the $-$ end of M, this divides Ω into two G-invariant subsets $\Omega^{\pm}$. There are three possibilities for each of $\Omega^{\pm}$, and the definition of the corresponding *end invariant* $\nu^{\pm}$ is as follows.

1. $\Omega^{\pm}$ is a topological disc, $\Omega^{\pm}/G$ is a once-punctured torus. In this case we define $\nu^{\pm}$ to be the Teichmüller parameter of $\Omega^{\pm}/G$. Manifold M_1 in figure 3 has both ends of this type.
2. $\Omega^{\pm}$ is an infinite union of discs, $\Omega^{\pm}/G$ is a triply-punctured sphere obtained from the boundary of $\Sigma \times (0,1)$ by deleting a curve $\gamma_{\pm}$. This curve corresponds to an element $W_{p/q}$ and in this case we define $\nu^{\pm} = p/q$ (see section 2.2.2). Manifold M_2 in figure 3 has one end of this type, and one end of the first type. Manifold M_3 has both ends of this type.
3. $\Omega^{\pm}$ is empty. This case can be considered a limit of cases where $\nu^{\pm} = p/q$ and we get $\nu^{\pm} \in \mathbb{R} - \mathbb{Q}$.

Writing $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} = S^1$, we define $\overline{\mathbb{H}} = \mathbb{H} \cup \hat{\mathbb{R}}$ (or equivalently $\overline{\mathbb{H}}$ is a closed disc), and Δ to be the diagonal of $\hat{\mathbb{R}} \times \hat{\mathbb{R}}$. For any representation $[\rho] \in \overline{\mathcal{QF}}$ we can assign its pair of end invariants $(\nu_-, \nu_+) \in (\overline{\mathbb{H}} \times \overline{\mathbb{H}}) - \Delta$. There is a continuous bijection

$$\nu^{-1} : (\overline{\mathbb{H}} \times \overline{\mathbb{H}}) - \Delta \longrightarrow \overline{\mathcal{QF}}$$

(but the inverse map ν is not even continuous). If a marked punctured torus group is in $\partial\mathcal{QF} = \overline{\mathcal{QF}} - \mathcal{QF}$ then either $\nu^{\pm} \in \mathbb{Q} \cup \{\infty\}$ corresponding to pinching a curve on $\Omega^{\pm}/G$ of slope $\nu^{\pm}$ to a point, or $\nu^{\pm}$ is an irrational real. See [Minsky99], [Bonahon86] and [Thurston80] for more details.

2.2 Combinatorics

2.2.1 Farey series

The Farey series and Farey graph are the foundation of the combinatorics of the Maskit slice, which we will define later. We define $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ (and similarly $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$). Rational numbers p/q and r/s satisfying $ps - rq = \pm 1$ are said to be *Farey neighbours*. For such fractions we define the operation of *Farey addition* $\oplus$ by

$$\frac{p}{q} \oplus \frac{r}{s} = \frac{p+r}{q+s}.$$

The Farey graph, embedded in the hyperbolic upper half plane model in figure 4, has vertex set $\hat{\mathbb{Q}}$, and two vertices are connected by an edge if they are neighbours. If p/q and r/s are neighbours, then p/q, r/s and $p/q \oplus r/s$ are the vertices of a triangle in the Farey graph. As mentioned in the introduction, cusp points correspond to rational numbers p/q. Two cusps whose corresponding rationals are Farey neighbours will be said to be *neighbouring cusps*.

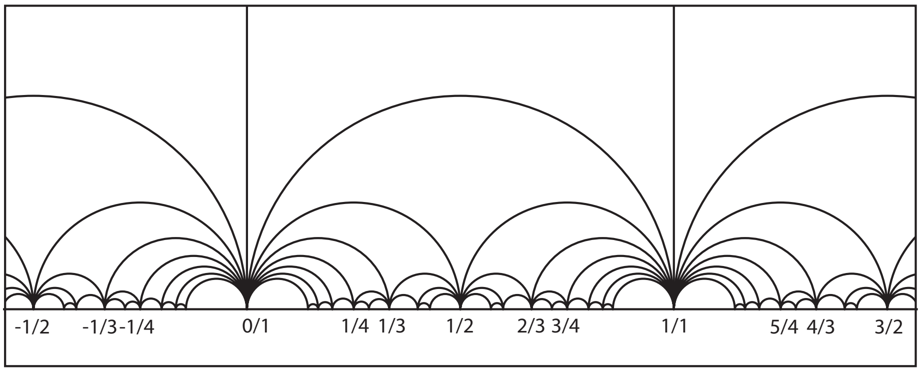

Figure 4: The Farey graph

2.2.2 The curve complex

We have already defined Σ to be a fixed once-punctured torus, now define S to be a fixed torus, say $\Sigma = S - \{*\}$. Now both $\pi_1(S)$ and $\pi_1(\Sigma)$ are generated by X and Y. The group $\pi_1(S)$ is the free Abelian group $\mathbb{Z}^2$, whereas, as we have noted, $\pi_1(\Sigma) = \langle X, Y \rangle$. Every free homotopy class of a simple closed curve on S or Σ is represented by an element of the fundamental group corresponding to an element of $\hat{\mathbb{Q}}$. On S, it is represented by the element $X^q Y^p$. On Σ, the same curve is represented by the word $W_{p/q}$ defined inductively below.

$$\begin{aligned}
W_{0/1} &= X \\
W_{1/0} &= Y \\
W_{-1/0} &= Y^{-1} \\
W_{p/q \oplus r/s} &= W_{p/q} W_{r/s} && \text{if } ps - rq = -1 \\
W_{p/q \oplus r/s} &= W_{r/s} W_{p/q} && \text{if } ps - rq = 1
\end{aligned}$$

See [Wright88], [KeenSeries93] and [Series85] for more details.

2.3 The Maskit slice

2.3.1 Definition

A *Maskit slice* (see [Maskit74]), which we will occasionally denote $\mathcal{M}$, is a slice of the boundary of $\mathcal{QF}$. It consists of elements $[\rho] \in \partial\mathcal{QF}$ such that the associated 3-manifold M has one end a fixed triply punctured sphere. In the notation of section 2.1.3, it is the set $\nu^{-1}(\{*\} \times \overline{\mathbb{H}})$ or $\nu^{-1}(\overline{\mathbb{H}} \times \{*\})$ where $* \in \hat{\mathbb{Q}}$. The interior of a Maskit slice corresponds to manifolds of type M_2 in figure 3, and the boundary of a Maskit slice corresponds to manifolds of type M_3.

It can be very simply embedded in $\mathbb{C}$ in the following way (see [Wright88]). Define

$$g : \mathbb{C} \longrightarrow \mathrm{Hom}(\Gamma, \mathrm{SL}_2\mathbb{C})$$

by

$$g(\mu)(X) = -i \begin{pmatrix} \mu & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad g(\mu)(Y) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Let $\pm : \mathrm{SL}_2\mathbb{C} \to \mathrm{PSL}_2\mathbb{C}$ be the quotient map. The map $[g] : \mathbb{C} \to \mathcal{R}_{\mathrm{p}}(\Gamma)$ is defined by setting $[g](\mu) = [\pm g(\mu)]$, and a detail of the set $\overline{\mathcal{M}} = [g]^{-1}(\overline{\mathcal{QF}}) \cap \mathbb{H}$ is shown in figure 1.

2.3.2 Trace functions

We define the trace function

$$\mathrm{Tr}_W : \mathcal{QF} \longrightarrow \mathbb{C}/\{\pm 1\}; \ [\rho] \longmapsto \mathrm{Tr}(\rho(W)),$$

where $W \in \Gamma$. On the Maskit slice, we define the p/q-trace function

$$\mathrm{Tr}_{p/q} : \mathbb{C} \longrightarrow \mathbb{C}; \ \mu \longmapsto \mathrm{Tr}(g(\mu)(W_{p/q})).$$

The following formula is due to David Wright, see [Wright88] and [MSW02]. We start from the formula for two matrices $M, N \in \mathrm{SL}_2\mathbb{C}$,

$$\mathrm{Tr}\, MN + \mathrm{Tr}\, M^{-1}N = \mathrm{Tr}\, M \cdot \mathrm{Tr}\, N.$$

If $p/q < r/s$, so that $W_{(p+r)/(q+s)} = W_{p/q}W_{r/s}$ (see section 2.2.2), then

$$\mathrm{Tr}\, W_{(p+r)/(q+s)} = \mathrm{Tr}\, W_{p/q} \cdot \mathrm{Tr}\, W_{r/s} - \mathrm{Tr}\, W_{p/q}^{-1}W_{r/s}.$$

It is relatively easy to show that $\mathrm{Tr}\, W_{(r-p)/(s-q)} = \mathrm{Tr}\, W_{p/q}^{-1}W_{r/s}$, and so

$$\mathrm{Tr}\, W_{(p+r)/(q+s)} = \mathrm{Tr}\, W_{p/q} \cdot \mathrm{Tr}\, W_{r/s} - \mathrm{Tr}\, W_{(r-p)/(s-q)}. \qquad (1)$$

Clearly $\mathrm{Tr}_{0/1}(\mu) = -i\mu$ and $\mathrm{Tr}_{1/0}(\mu) = 2$, and so equation 1 shows that $\mathrm{Tr}_{p/q}(\mu)$ is a polynomial in μ (which it is fairly easy to see has degree q).

2.3.3 Cusp structure and coordinates

There is a unique point $\mu_{p/q}$ on the boundary $\partial\mathcal{M}$ of the Maskit slice such that $\mathrm{Tr}_{p/q}(\mu_{p/q}) = 2$. This point is called the p/q-cusp.

Figure 5 gives three different views on cusps. The upper left picture shows the boundary of the Maskit slice in a neighbourhood of a cusp (in fact the 9/17-cusp shown in more detail in figure 6). It is labelled "μ-coord" because μ is the variable in the definition of the embedding of the Maskit slice in $\mathbb{C}$ in section 2.3.1. The upper right hand picture shows the same portion of the boundary after the trace function corresponding to that cusp ($\mathrm{Tr}_{9/17}(\mu)$) has been applied to it. The lower picture shows the slice in the coordinates defined by the end invariant of the non-fixed end of the corresponding manifold. This is labelled the "ν-coord" because the end invariants defined in section 2.1.3 are written $\nu^{\pm}$.

In the upper left hand figure and the upper right hand figure, you can see that the boundary is locally "cuspidal", that is it reaches a sharp point. Miyachi showed in [Miyachi03] that the boundary is approximately $(2, 3)$-cuspidal in a neighbourhood of a cusp. That is, they are locally approximately like a scaled, rotated and translated copy of the curve $x^2 = y^3$.

Write $\mu \in \mathbb{C}$ for a point in the embedding of the Maskit slice, so that μ is in the coordinates of the upper left picture of figure 5. Define

$$\tau = \frac{1}{q^2(\nu - p/q)},$$

where ν is the end invariant of the non-fixed end of the corresponding manifold (that is, the coordinate in the lower picture). We show in [Goodman06] that in the trace coordinates (the upper right picture),

$$\mathrm{Tr}_{p/q}(\mu) = 2 - \frac{\pi^2}{\tau^2} + O(\tau^{-3}).$$

In the μ-coordinate,

$$\mu = \mu_{p/q} - \frac{\pi^2}{\tau^2\, \mathrm{Tr}'_{p/q}(\mu_{p/q})} + O(\tau^{-3}).$$

It is proved in [Miyachi03], and more generally in [ChoiSeries06], that $\mathrm{Tr}'_{p/q}(\mu_{p/q}) \neq 0$. It follows that in a sufficiently small neighbourhood of $\mu_{p/q}$, $\mathrm{Tr}_{p/q}(\mu)$ is a coordinate for the Maskit slice. In section 3.1 below we make a conjecture about how large a neighbourhood of $\mu_{p/q}$ this is a coordinate, and this conjecture turns out to be the key to showing that the boundary of the Maskit slice is twisting almost everywhere.

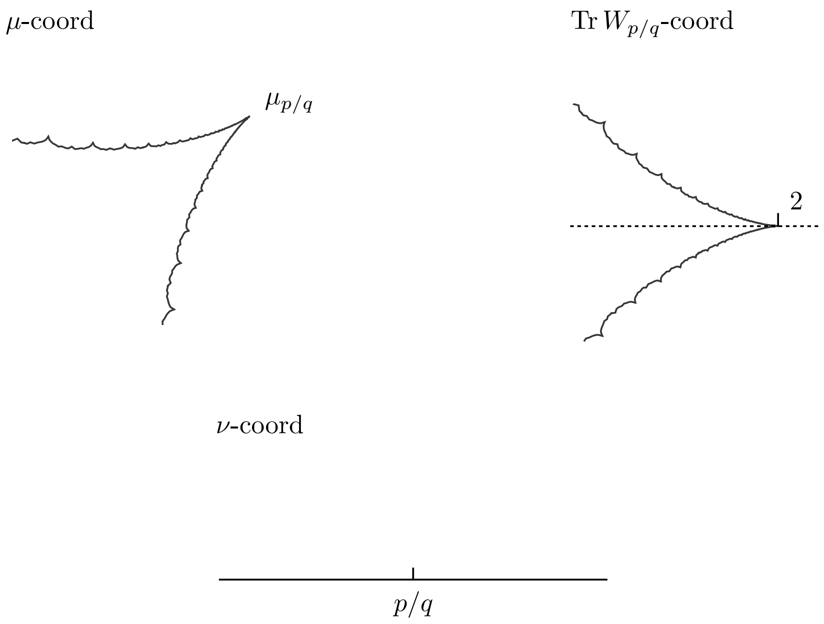

Figure 5: Three views on cusps

2.4 Twisting

Straight from thesis

Let $\mathbb{D} \subseteq \mathbb{C}$ be the unit disc in $\mathbb{C}$ and $f : \mathbb{D} \to G \subseteq \mathbb{C}$ conformal. We define two sorts of behaviour at the boundary.

A *Stolz angle* at $\zeta \in \partial\mathbb{D}$ is any set of points $\Delta \subseteq \mathbb{D}$ a bounded hyperbolic distance from the radius $[0, \zeta]$. We say f has the *angular limit* $a \in \hat{\mathbb{C}}$ at $\zeta \in \partial\mathbb{D}$ if

$$\lim_{z \to \zeta, z \in \Delta} f(z) = a \tag{2}$$

for every Stolz angle Δ at ζ. We will write $f(\zeta)$ for the angular limit a. We say f is *conformal* at $\zeta \in \partial\mathbb{D}$ if

$$f'(\zeta) = \lim_{z \to \zeta, z \in \Delta} \frac{f(z) - f(\zeta)}{z - \zeta} = \lim_{z \to \zeta, z \in \Delta} f'(z) \neq 0, \infty \tag{3}$$

for every Stolz angle Δ at ζ. We say that f is *twisting* at ζ if the angular limit $f(\zeta) \neq \infty$ exists and

$$\liminf_{z \to \zeta, z \in \Gamma} \arg(f(z) - f(\zeta)) = -\infty, \quad \limsup_{z \to \zeta, z \in \Gamma} \arg(f(z) - f(\zeta)) = +\infty$$

for every curve $\Gamma \subseteq \mathbb{D}$ ending at ζ. We say that $f(\zeta)$ is *sectorially accessible* from G if G contains an open triangle with vertex $f(\zeta)$. We define $\operatorname{Sect}(f)$ to be the set of all $\zeta \in \partial\mathbb{D}$ such that $f(\zeta)$ is sectorially accessible. Clearly if $f(\zeta)$ is sectorially accessible it cannot be twisting. We write λ for linear measure (one dimensional Lebesgue measure) on subsets of $\mathbb{C}$.

Essentially (that is, up to sets of measure 0) we think of sectorially accessible and conformal as the same, and the opposite of twisting. This is the intuitive content of theorem 2.4.1 below. See [McMillan69] and chapter 6 of [Pomm92].

Theorem 2.4.1 (McMillan Twist Theorem) *At almost all $\zeta \in \partial\mathbb{D}$ the map f is either conformal or twisting.*

The twist theorem implies that $\operatorname{Sect}(f)$ differs from the set of conformal boundary points by a set of measure 0.

2.5 Continued fractions

Straight from thesis

We need some results about continued fractions. These follow quite simply by applying the ergodic theorem to the Gauss measure and the continued fraction transformation.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a *probability space* and $T : \Omega \to \Omega$ a *measure preserving transformation.* That is, $\mathcal{F}$ is a σ-algebra of subsets of Ω, $\mathbb{P}$ is a probability measure on $\mathcal{F}$, and T is a measurable function satisfying $\mathbb{P}(T^{-1}A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. A set $A \in \mathcal{F}$ is *invariant* if $T^{-1}A = A$. A measurable function g on Ω is said to be *invariant* if $g(T\omega) = g(\omega)$ a.e. The transformation T is *ergodic* if every invariant set has measure 0 or 1. If $f : \Omega \to \mathbb{R}$ is integrable we define the *expectation* of f to be $\mathbb{E}[f] = \int f \, \mathrm{d}\mathbb{P}$.

Theorem 2.5.1 (The Ergodic Theorem) *If f is integrable, then there exists an integrable, invariant function $\hat{f}$ such that $\mathbb{E}[\hat{f}] = \mathbb{E}[f]$ and*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k\omega) = \hat{f}(\omega) \ \text{a.e.}$$

If T is ergodic then $\hat{f}(\omega) = \mathbb{E}[f]$ a.e.

For a proof of the ergodic theorem, see [Bill65]. As a simple application we get the following.

Corollary 2.5.2 *Let T be ergodic and $f = I_A$ the indicator function of A. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I_A(T^k\omega) = \mathbb{P}(A) \ \text{a.e.}$$

We apply this to prove several results about continued fractions. We consider the unit interval $[0, 1]$ with the *Gauss measure* defined by

$$\mathbb{P}(A) = \frac{1}{\log 2} \int_A \frac{\mathrm{d}x}{1 + x}.$$

This measure is absolutely continuous with respect to Lebesgue measure so the terms integrable and a.e. apply equally to the Gauss measure or Lebesgue measure. We define the function

$$T(\omega) = \begin{cases} \{1/\omega\} & \text{if } \omega \neq 0, \\ 0 & \text{if } \omega = 0. \end{cases}$$

Here $\{x\}$ means the fractional part of x, and $[x]$ means the integer part. This function T is ergodic with respect to the Gauss measure, see [Bill65]. We also define the *partial quotients*

$$a(\omega) = \begin{cases} [1/\omega] & \text{if } \omega \neq 0, \\ \infty & \text{if } \omega = 0 \end{cases}$$

and

$$a_n(\omega) = a(T^{n-1}\omega).$$

We use the notation $[a_1 \; a_2 \; a_3 \; ...]$ to mean the continued fraction

$$\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}}.$$

We define $p_n(\omega)$ and $q_n(\omega)$ to be the numerator and denominator of $[a_1 \dots a_n]$. We get the following relations for $n \geq 1$,

$$\begin{aligned} p_{n+1} &= a_{n+1} p_n + p_{n-1} \\ q_{n+1} &= a_{n+1} q_n + q_{n-1} \end{aligned} \tag{4}$$

If we specify $p_{-1}/q_{-1} = 1/0$ and $p_0/q_0 = 0/1$ these relations can be used to define p_n and q_n. From this, it is straightforward to prove (see [Bill65] chapter 1, section 4) that

$$\frac{1}{q_n(q_n + q_{n+1})} \leq \left| \omega - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}.$$

From this, it is easy to prove the following result, which we will use later,

$$\frac{1}{(a_{n+2}+1)(a_{n+1}+1)+1} \leq \left|\omega - \frac{p_{n+1}}{q_{n+1}}\right| \Big/ \left|\omega - \frac{p_n}{q_n}\right| \leq \frac{a_{n+1}+2}{a_{n+1}^2 a_{n+2}}. \qquad (5)$$

Let f be the indicator of the set $\{\omega : a_1(\omega) = k\}$. Applying corollary 2.5.2 we see that the asymptotic relative frequency of k among the partial quotients is

$$\frac{1}{\log 2}\int_{1/(k+1)}^{1/k} \frac{\mathrm{d}x}{1+x} = \frac{1}{\log 2}\log\frac{(k+1)^2}{k(k+2)}.$$

In particular, since the right hand side is nonzero for all k, k appears infinitely often in the continued fraction expansion of almost all ω. This also shows that the partial quotients are unbounded for almost all ω.

By a similar method, we can find the asymptotic relative frequency of the sequence $k_1, k_2, \ldots, k_m$. We let f be the indicator of the set

$$A(k_1, \ldots, k_m) = \{\omega : a_1(\omega) = k_1, \ldots, a_m(\omega) = k_m\},$$

and apply corollary 2.5.2 to get the asymptotic relative frequency of the sequence to be

$$\mathbb{P}(A(k_1, \ldots, k_m)) = \frac{1}{\log 2}\int_{A(k_1,\ldots,k_m)} \frac{\mathrm{d}x}{1+x}.$$

In fact, all we need here is that this probability and asymptotic relative frequency is nonzero.

Lemma 2.5.3 *Every sequence $(k_1, \ldots, k_m)$ occurs infinitely often in the continued fraction expansion of almost all ω.*

The following results are also proved in [Bill65] by relatively simple applications of the ergodic theorem.

$$\lim_{n\to\infty} \frac{1}{n}\log q_n(\omega) = \frac{\pi^2}{12\log 2} \quad \text{a.e.}$$

Roughly, we think of this as saying that for large n and almost all ω,

$$q_n(\omega) \sim \beta^n,$$

where $\beta = e^{\frac{\pi^2}{12\log 2}} \approx 3.28$. It also follows that

$$\lim_{n\to\infty} \frac{1}{n}\log\left|\omega - \frac{p_n(\omega)}{q_n(\omega)}\right| = -\frac{\pi^2}{6\log 2} \quad \text{a.e.}$$

Again, we think of this as saying

$$\left|\omega - \frac{p_n(\omega)}{q_n(\omega)}\right| \sim \gamma^n,$$

where $\gamma = e^{-\frac{\pi^2}{6\log 2}} \approx 0.09$.

We also have results on Diophantine approximation. It is possible to prove the following directly.

Lemma 2.5.4 *For all irrational $\omega \in [0,1]$, we have*

$$\left|\omega - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}.$$

We do not use the rest of these results directly, but they are important results in the theory of continued fractions and Diophantine approximation which help to give a better intuitive picture of what is going on.

Theorem 2.5.5 *The event $a_n(\omega) > \alpha_n$ occurs infinitely often with probability 0 if $\sum 1/\alpha_n$ converges and probability 1 if it diverges.*

Theorem 2.5.6 *Let $f(q) : \mathbb{N} \to \mathbb{R}^+$.*

1. *If $qf(q)$ is nonincreasing and $\sum f(q) = \infty$ then for almost all ω we have*

$$\left|\omega - \frac{p}{q}\right| < \frac{f(q)}{q}$$

for infinitely many p and q.

2. *If $\sum f(q) < \infty$ then for almost all ω the inequality holds for only finitely many p and q.*

In particular, setting $f(q) = \frac{1}{q \log q \log \log q}$ and $f(q) = \frac{1}{q(\log q)^2}$, whose sums diverge and converge respectively, we have the following.

Corollary 2.5.7 *For almost all ω, for infinitely many p and q we have*

$$\frac{1}{q^2(\log q)^2} < \left|\omega - \frac{p}{q}\right| < \frac{1}{q^2 \log q \log \log q}.$$

To show these sums converge or diverge we apply Cauchy's integral test which states in this case that $\sum^\infty f(q)$ converges or diverges if $\int^\infty f(q)\mathrm{d}q$ converges or diverges. The sum of $1/q \log q$ diverges because the integral of $1/q \log q$ is $\log \log q$ which tends to infinity as $q \to \infty$. Substituting $u = \log q$ we see that the integral of $1/q \log q \log \log q$ is the integral of $1/u \log u$ and so also diverges. Similarly, the sum of $1/q(\log q)^2$ converges because the integral is $-1/\log q$ which tends to 0 as $q \to \infty$.

For more details on continued fractions, see [Bill65] and [HarWri79].

3 Conjectures and evidence

We have proved that the boundaries of the Maskit and Bers slices spiral infinitely at an uncountable, dense set. However, there is very good numerical evidence to suggest that in fact they spiral infinitely at almost every point in their boundaries. We present an argument for this based on two unproven conjectures (sections 3.1 and 3.2). Our argument relies on some facts from complex analysis (section 2.4) and number theory (section 2.5).

Throughout this chapter and the next, we refer to C++ and Mathematica files and functions. These are included on CD attached to this thesis. They are also included on the author's web page, which at the time of writing is maths.thesamovar.net.

3.1 Uniform local coordinates

Our argument in section 3.3.2 requires a somewhat technical result. Consider the following situation. Take the p/q-cusp in the Maskit slice, and let μ be the usual coordinates, and $t = \mathrm{Tr}_{p/q}(\mu)$ be the trace coordinates in a small neighbourhood. You can see in figure 6 that the linear part of $\mathrm{Tr}_{p/q}(\mu)$ is quite a good estimate of $\mathrm{Tr}_{p/q}(\mu)$ in a small region surrounding the cusp $\mu_{p/q}$. We have good numerical evidence to support the following conjecture which is an attempt to capture this.

Conjecture 3.1.1 (Uniform Local Coordinates) *There exists $\epsilon > 0$ such that for all p/q, for the region $|t - 2| \leq \epsilon$, we have*

$$\frac{1}{2}\frac{|t-2|}{|\,\mathrm{Tr}'_{p/q}(\mu_{p/q})|} \leq |\mu - \mu_{p/q}| \leq \frac{3}{2}\frac{|t-2|}{|\,\mathrm{Tr}'_{p/q}(\mu_{p/q})|}.$$

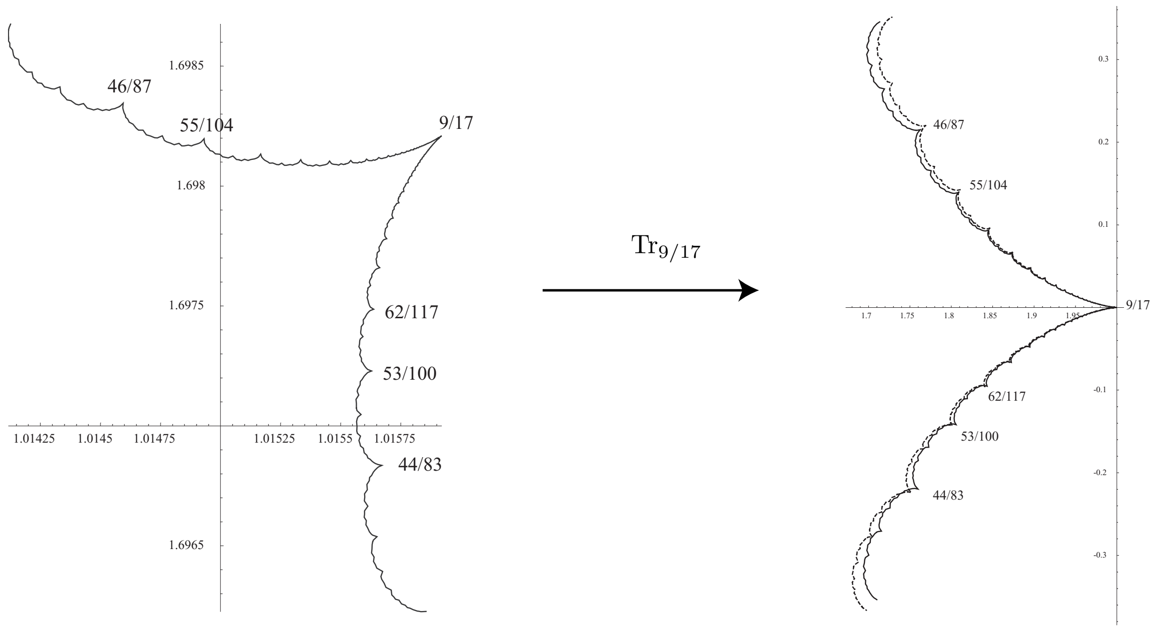

Figure 6: Trace coordinates. The figure on the left hand side is a small region in the Maskit slice surrounding the 9/17-cusp. The solid line on the figure on the right hand side is the same region in the $\mathrm{Tr}_{9/17}$ trace coordinates. The dashed line is the same region if you only apply the linear part of the function $\mathrm{Tr}_{9/17}$. The fractions on both figures correspond to the cusps.

The purpose of this conjecture will become clear in section 3.3.2. However, roughly speaking it says that if you can estimate $|\mathrm{Tr}'_{p/q}(\mu_{p/q})|$ and $\mathrm{Tr}_{p/q}(\mu)$ then you can estimate $|\mu - \mu_{p/q}|$. Not having this conjecture or something similar is a surprisingly large barrier to proving results about the Maskit slice. For example, Lemma 5 in [Goodman06] tells us that the sequence of neighbouring cusps $\mu_n \to \mu_\infty = \mu_{p/q}$ satisfies

$$\mu_n = \mu_\infty - \frac{\pi^2}{n^2 \mathrm{Tr}'_\infty(\mu_\infty)} + O(n^{-3}).$$

The problem is that the implicit constant in the $O(n^{-3})$ term depends on p/q in a complicated way. However, if we write $t_n = \mathrm{Tr}_{p/q}(\mu_n)$ then we know that

$$t_n = 2 - \frac{\pi^2}{n^2} + O(n^{-3}),$$

where the $O(n^{-3})$ term depends on universal constants independent of p/q. If the conjecture is correct then we can say that

$$|\mu_n - \mu_\infty| = \Theta\left(\frac{\pi^2}{n^2|\mathrm{Tr}'_\infty(\mu_\infty)|}\right),$$

where the implicit constants are universal (don't depend on p/q). Note that the $\Theta(-)$ notation here is similar to the $O(-)$ order notation. Roughly speaking $f = O(g)$ means that f is of the order of g or smaller, whilst $f = \Theta(g)$ means that f is of the same order as g.

Don't have an appendix here, so define the Θ notation here or rewrite

The numerical evidence for this conjecture is as follows. Assume for the moment that $\mathrm{Tr}_{p/q}$ is invertible in the region we are considering. Let

$$E_{p/q}(t) = \mu - \mu_{p/q} - \frac{t-2}{\mathrm{Tr}'_{p/q}(\mu_{p/q})}$$

be the error term (consisting of the second and higher order terms in the series expansion of $\mathrm{Tr}^{-1}_{p/q}$). We want to show that

$$|E_{p/q}(t)| \leq \frac{|t-2|}{2|\mathrm{Tr}'_{p/q}(\mu_{p/q})|}.$$

So write

$$\frac{\mathrm{Tr}'_{p/q}(\mu_{p/q})E_{p/q}(t)}{t-2} = \sum_{n=2}^{\infty} a_n(t-2)^{n-1}.$$

Now if $|t-2| \leq 1$ then this sum will be less than $\sum_{n=2}^{\infty} |a_n|$. So if this sum is always less than 1/2 the conjecture follows. Numerical evidence suggests that this is the case. This evidence only includes cases up to $q \leq 20$ because of numerical stability problems with Mathematica.

An alternative way of showing numerically that $E_{p/q}(t)$ is bounded in this way is to use Cauchy's theorem

$$f^{(k)}(z) = \frac{k!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{k+1}}\mathrm{d}\zeta.$$

Suppose C is a circle about z of radius ϵ and $|f(z)| \leq A$ on C, then

$$\left|\frac{f^{(k)}(z)}{k!}\right| \leq \frac{A}{\epsilon^k}.$$

Now suppose that f in the above is defined by $f(z) = \mathrm{Tr}^{-1}_{p/q}(t) - \mu_{p/q}$ and $z = t-2$. If we could find a constant B not depending on p/q such that on C we have $|f(z)| \leq B/|\mathrm{Tr}'_{p/q}(\mu_{p/q})|$ then the conjecture follows. For $k \geq 2$, the quantity $f^{(k)}(0)/k!$ is the kth coefficient of the Taylor series of $E_{p/q}(t)$. It follows that when $|t-2| \leq \eta\epsilon$

$$\left|\frac{\mathrm{Tr}'_{p/q}(\mu_{p/q})E_{p/q}(t)}{t-2}\right| \leq \sum_{k=2}^{\infty} \frac{A|\mathrm{Tr}'_{p/q}(\mu_{p/q})|\eta^{k-1}\epsilon^{k-1}}{\epsilon^k} = \frac{B\eta}{\epsilon(1-\eta)}.$$

By choosing η sufficiently small we can ensure that this is always less than 1/2.

The numerical evidence for the existence of this universal bound is very good indeed. It is difficult to compute $\mathrm{Tr}^{-1}_{p/q}$ so we take the following approach (see figure 7). For every p/q let the circle D be the circle of radius $1/|\mathrm{Tr}'_{p/q}(\mu_{p/q})|$ around $\mu_{p/q}$. It turns out that for all $q \leq 1000$ the image under $\mathrm{Tr}_{p/q}$ of D is always contained in an annulus which seems to have universal inner and outer radii (about 0.87 and 1.15 respectively). This analysis is carried out by the C++ function view::imagecircleanalysis in the file maskitalgorithms.cpp. Since $f'(z) = (\mathrm{Tr}^{-1}_{p/q})'(t) = 1/\mathrm{Tr}'_{p/q}(\mu)$, $f'(z) \neq 0$ and so f cannot have a local maximum within C, the circle of radius 0.87 about $t = 2$. This implies the existence of a universal bound of the required type (by the maximum modulus principle).

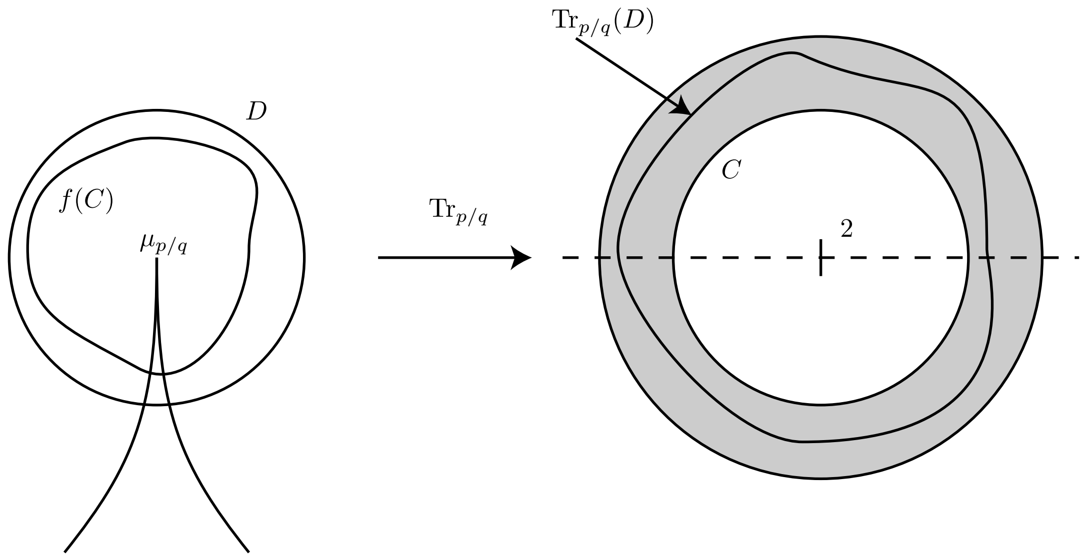

Figure 7: Showing numerically that $\mathrm{Tr}_{p/q}$ is approximately linear

Finally, we note that numerical evidence suggests that on the disc E of radius $1/|\mathrm{Tr}'_{p/q}(\mu_{p/q})|$ about $\mu_{p/q}$, the function $\mathrm{Tr}_{p/q}$ is invertible. Firstly, a small number of sample calculations suggest that the image of E is always simply connected. We provide no numerical evidence in support of this, other than to say that in a small number of test cases it seemed to be the case, and it is very implausible that it is not the case. However, more extensive computations (for all $q \leq 80$) suggest strongly that on E, $\mathrm{Tr}'_{p/q}(\mu) \neq 0$. By analytic continuation, this would imply that $\mathrm{Tr}_{p/q}$ is invertible on E. See the Mathematica notebook invertibility-regions.nb for these calculations.

In conclusion, the evidence for this conjecture is far from complete (in particular, because the argument above relies on the invertibility of $\mathrm{Tr}_{p/q}$ in the required region, for which the evidence is relatively weak), but it is nonetheless strong enough to make it likely to be true.

Recursive trace formula and Farey addition need a reference or something included earlier

Probably the best method to try to prove this conjecture would be to first prove some statement about the trace functions for all p/q using induction on the level of p/q and the formula $\mathrm{Tr}\, W_{(p+r)/(q+s)} = \mathrm{Tr}\, W_{p/q} \cdot \mathrm{Tr}\, W_{r/s} - \mathrm{Tr}\, W_{(r-p)/(s-q)}$. The level of p/q is the number of steps needed to get to p/q using Farey addition. The second step would be to derive the conjecture from this statement about the trace functions for all p/q. The proof would probably be relatively straightforward, the difficult thing is working out precisely what statement to prove using induction.

3.2 Trace derivatives

The argument in section 3.3.2 also requires one more result, conjecture 3.2.3 below. However, this section also allows us to make a conjecture about the Hausdorff dimension of the Maskit slice in section 3.4.

Define $T_{p/q} = |\,\mathrm{Tr}'_{p/q}(\mu_{p/q})|$. Given $\omega \in [0,1]$ and $p_n/q_n \to \omega$ the partial approximants, define $T_n = T_n(\omega) = T_{p_n/q_n}$. We make several conjectures about the statistical and limiting properties of $T_{p/q}$ on the basis of numerical evidence. See the Mathematica notebook trace-derivatives.nb.

We first argue that for almost all ω, for large n, $\log T_n / \log q_n \in [1.69846, 2.35407]$. First, we note that there is strong statistical evidence to suggest that for all ω and for all n, $T_n/T_{n-1} \in [T_{1/a_n}, T_{1/a_n+1}]$. The following table gives the minimum and maximum values of T_n/T_{n-1} for 30000 samples subject to the restriction that $q_n \leq 2000$.

$a_n(\omega)$	$\min T_n/T_{n-1}$	$\max T_n/T_{n-1}$	T_{1/a_n}	T_{1/a_n+1}	Samples
1	1.	3.4641	1.	3.4641	30718
2	3.4641	6.97359	3.4641	6.97359	12174
3	6.97359	11.875	6.97359	11.875	6666
4	11.875	18.9384	11.875	18.9384	4084
5	18.9384	28.9523	18.9384	28.9523	2872
6	28.9523	42.6073	28.9523	42.6073	2098
7	42.6073	60.5356	42.6073	60.5356	1605
8	60.5356	83.3487	60.5356	83.3487	1239
9	83.3487	111.652	83.3487	111.652	1025
10	111.652	146.05	111.652	146.05	778
11	146.05	187.148	146.05	187.148	713
12	187.148	235.55	187.148	235.55	549
13	235.55	291.864	235.55	291.864	508
14	291.864	356.695	291.864	356.695	459
15	356.695	430.651	356.695	430.651	382
16	430.651	514.337	430.651	514.337	294
17	514.337	608.363	514.337	608.363	320
18	608.363	713.334	608.363	713.334	288
19	713.334	829.858	713.334	829.858	232
20	829.858	958.543	829.858	958.543	225

As noted, this suggests the following conjecture.

Conjecture 3.2.1 *For all ω, n, we have that $|\,\mathrm{Tr}'_n\,|/|\,\mathrm{Tr}'_{n-1}\,| \in [\mathrm{Tr}'_{1/a_n}, \mathrm{Tr}'_{1/a_n+1}\,|]$.*

Now, the event $a_n(\omega) = m$ occurs with relative frequency

$$p_m = \frac{1}{\log 2} \log \frac{(m+1)^2}{m(m+2)}.$$

Consider the first n partial quotients of ω. In the limit, np_m of these will be m. Therefore, very roughly speaking, for almost all ω, in the limit we will have

$$\prod_{m=1}^{\infty} T_{1/m}^{np_m} \leq T_n \leq \prod_{m=1}^{\infty} T_{1/(m+1)}^{np_m}.$$

Taking logs and dividing by n we get

$$\sum_{m=1}^{\infty} p_m \log T_{1/m} \leq \log T_n/n \leq \sum_{m=1}^{\infty} p_m \log T_{1/(m+1)}.$$

Evaluating these sums numerically using the values of $T_{1/m}$ from $m = 1$ to $m = 2000$ gives $\log T_n/n \in [2.01533, 2.79327]$. We already know that $q_n \sim \beta^n$ where $\log \beta = \pi^2/12 \log 2$. Combining these gives the limiting value of $\log T_n/\log q_n \in [1.69846, 2.35407]$ for almost all ω.

Evaluating $\log T_{p/q}/\log q$ for all p/q with $q \leq 2000$ gives the histogram in figure 8. As you can see, there are many values outside the limiting range $[1.69846, 2.35407]$, but that the majority are within this range, which is what we would expect. The probability p_m is only a limiting relative frequency, therefore it is possible that for any particular n, $\log T_n/\log q_n$ might be outside these bounds. The constant term introduced into the size of T_n will vanish in the limit when we take logs and divide by $\log q_n$ (which tends to infinity).

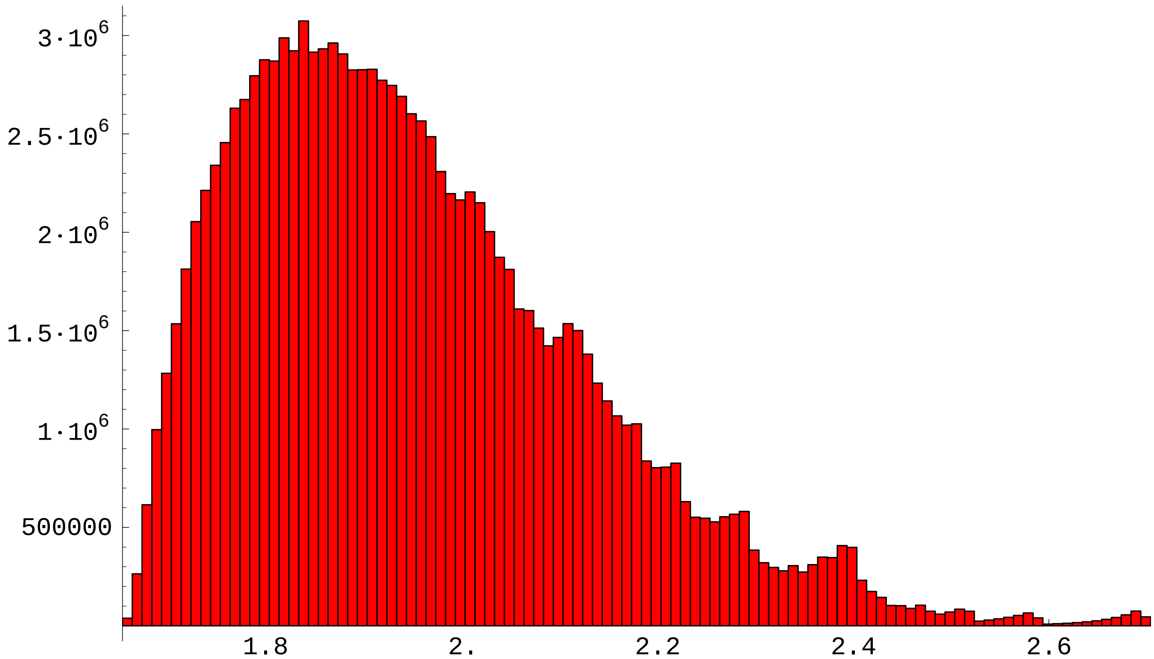

Figure 8: Histogram of values of $\log T_{p/q}/\log q$ for $q \leq 2000$

Conjecture 3.2.2 *Let $p_n/q_n \to \omega \in [0, 1]$ be the sequence of partial approximants. Let $T_n = T_{p_n/q_n}$. Then for almost all ω, we have*

$$\liminf_n \frac{\log T_n}{\log q_n} \geq 1.69$$

and

$$\limsup_n \frac{\log T_n}{\log q_n} \leq 2.36.$$

It may even be possible that something considerably stronger than this is true. If we choose $\omega \in [0, 1]$ uniformly at random and plot $\log T_n$ against $\log q_n$ we see that all of these graphs are very close to being straight lines. Figure 9 shows these graphs for nine randomly selected ω up to $q \leq 2000$.

The shaded area is the region $\log T_n / \log q_n \in [1.69846, 2.35407]$, and the black line is the best fit. The first conjecture states that eventually all the points will be contained in this shaded area, the stronger conjecture says that eventually the points will tend toward being a straight line contained in this area. This stronger conjecture is equivalent to saying that

$$\liminf_n \frac{\log T_n}{\log q_n} = \limsup_n \frac{\log T_n}{\log q_n}.$$

Equivalently, $T_n \sim q_n^\alpha$ for some $\alpha \in [1.69846, 2.35407]$.

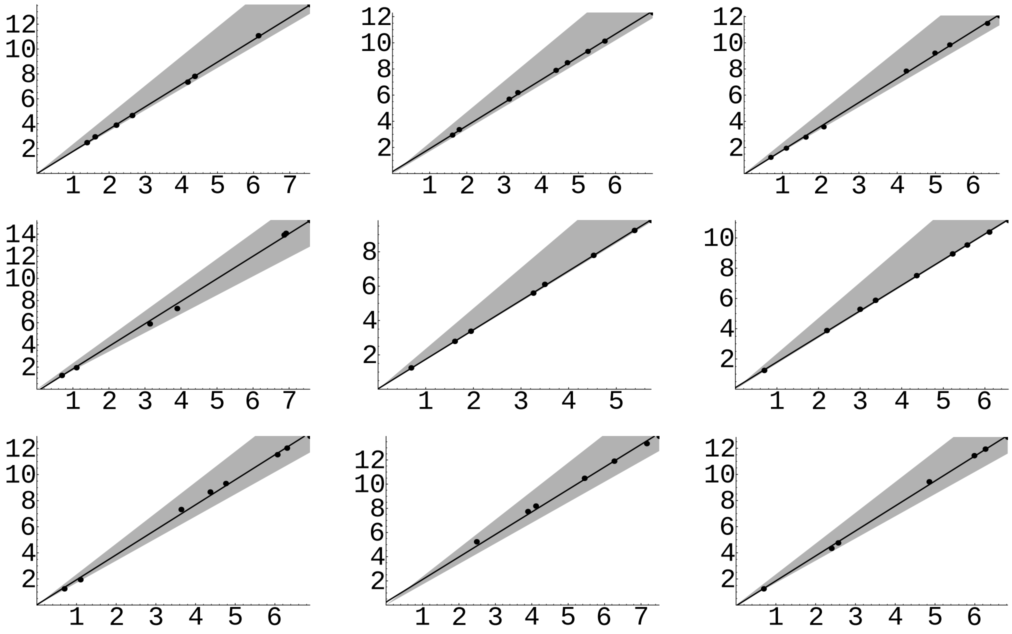

Figure 9: Nine graphs of $\log T_n$ against $\log q_n$

Running a linear regression on 10000 randomly chosen ω gives us a correlation coefficient of $r^2 \geq 0.985$ with an average r^2 of 0.998. The estimated variance was at most 0.47 and the average estimated variance was 0.045. These numbers are very good. Unfortunately, these linear regressions could only be applied to graphs with at most 15 points because of the requirement that $q \leq 2000$. This evidence is considerably less convincing than the previous evidence.

More likely is that almost all the time $\log T_n / \log q_n$ varies in the interval $[1.69846, 2.35407]$, mostly tending towards the bottom end when the common low integers appear in the continued fraction expansion, with occasional large jumps towards the top end when the infrequent large integers appear.

We could define the *exponent distribution interval* E_ω of $\omega \in [0, 1]$ to be the interval

$$E_\omega = [\liminf_n \frac{\log T_n}{\log q_n}, \limsup_n \frac{\log T_n}{\log q_n}].$$

The first conjecture states that for almost all ω, $E_\omega \subseteq [1.69846, 2.35407]$, the second states that for almost all ω, $|E_\omega| = 1$. Numerical evidence suggests strongly that for all ω, $E_\omega \subseteq [1.6, 3]$. Even stronger than this last statement, the numerical evidence suggests the following conjecture.

Conjecture 3.2.3 *For all p/q, $q^{1.6} \leq |\operatorname{Tr}'_{p/q}| \leq q^3$.*

3.3 Spiralling almost everywhere

We propose the following conjecture.

Conjecture 3.3.1 *The boundary of the Maskit slice is twisting almost everywhere.*

There are two ways of seeing why this is probably true. The first is more intuitively suggestive, but less workable. The second is less intuitive but only requires the very plausible conjectures 3.1.1 and 3.2.3 to make it work. We also suspect that this result is likely to be true for other slices of quasifuchsian space, such as the Bers and Earle slices.

refs for Bers, Earle slices?

3.3.1 Random walk

Either define spiralling degree here or earlier and include a section ref

Define the function

$$\theta : [0,1] \times \mathbb{N} \longrightarrow \mathbb{R}; (\omega, n) \mapsto \text{sp.deg}\, \wp_{p_n(\omega)/q_n(\omega)}$$

sending (ω, n) to the spiralling degree of the pleating ray of the nth partial approximant of ω. Essentially, the proof of Theorem 1 of [Goodman06] shows that for certain ω, $\theta(\omega, n) \to \infty$ as $n \to \infty$. The proof of Theorem 2 of the same paper shows that for these ω, if $\theta(\omega, n) \to \infty$ then the boundary is spiralling infinitely at $f(\omega)$. Although we have not proved it, it seems intuitively plausible that for any ω, if $\sup_n \theta(\omega, n) = +\infty$ and $\inf_n \theta(\omega, n) = -\infty$ then the boundary is twisting at $f(\omega)$.

Define the random variable Θ_n on the state space $[0,1]$ with either Lebesgue measure or Gauss measure by $\Theta_n(\omega) = \theta(\omega, n)$. If it were the case that $\Delta\Theta_n := \Theta_{n+1} - \Theta_n$ were independent, identically distributed random variables (distributed like $\Delta\Theta$ say), then it would almost immediately follow that the boundary is twisting almost everywhere. As long as $\mathbb{P}(\Delta\Theta > \epsilon) > 0$ and $\mathbb{P}(\Delta\Theta < -\epsilon) > 0$ for some ϵ, then the Θ_n would be a nontrivial one dimensional random walk and would therefore get close to $\pm\infty$ infinitely often.

Unfortunately, there are two problems. Firstly, the $\Delta\Theta_n$ are not independent. Secondly, we would need to prove that $\sup_n \theta(\omega, n) = +\infty$ and $\inf_n \theta(\omega, n) = -\infty$ together imply that the boundary is twisting at $f(\omega)$. The second problem is probably not too difficult, but the first is. Even though the $\Delta\Theta_n$ are not independent, it seems on the basis of numerical work that they are sufficiently close to independent and identically distributed that the conclusion should be true.

eq ref for trace derivative formula or list the equation here

If we could show that there was a uniform N such that whenever the nth partial approximant satisfied $a_n(\omega) > N$ we had $|\Delta\Theta_n(\omega)| > \epsilon$, then this would be enough. Unfortunately, the trace derivative formula involves terms that depend crucially on $p_n(\omega)/q_n(\omega)$, and so finding such a uniform N cannot be guaranteed. Again, numerically it seems as though quite small choices of N do in fact suffice ($N = 3$ or $N = 4$ seems to work).

The numerical basis for these claims is as follows. The program (Mathematica notebook randomwalk.nb) was instructed to randomly choose 30000 different $\omega \in [0,1]$ (with a uniform distribution). For each ω, the program runs through the continued fraction expansion as far as numerical precision allows and with the restriction that $q_n(\omega) < 2000$. Whenever it encounters $a_n(\omega) = N$ it stores the value of $|\Delta\Theta_n(\omega)|$. This process was repeated for various values of N between 1 and 20, and the minimum and maximum values for $|\Delta\Theta_n(\omega)|$ are summarised in the second and third columns of the table below. The fifth and sixth columns summarise the same data for the case $a_n(\omega) \geq N$. Note that the maximum values in the sixth column are close to the maximum possible value of $\pi/2 \approx 1.57$. Also note that the difference between the second and third columns gets quite small as N increases, which gives some plausibility to the idea that the $\Delta\Theta_n$ are approximately identically distributed for large $a_n(\omega)$.

N	$\min_= \Delta\Theta_n(\omega)$	$\max_= \Delta\Theta_n(\omega)$	Samples	$\min_\geq \Delta\Theta_n(\omega)$	$\max_\geq \Delta\Theta_n(\omega)$	Samples
1	0	0.229895	30800	0	1.56769	71787
2	7.4455×10^{-10}	0.151893	12125	7.4455×10^{-10}	1.56769	40987
3	0.0519555	0.335039	6633	0.0519555	1.56769	28862
4	0.259119	0.512935	4081	0.259119	1.56769	22229
5	0.449701	0.662867	2793	0.449701	1.56769	18148
6	0.606954	0.782329	2058	0.606954	1.56769	15355
7	0.731879	0.877532	1609	0.731879	1.56769	13297
8	0.830784	0.953197	1234	0.830784	1.56769	11688
9	0.910019	1.01443	972	0.910019	1.56769	10454
10	0.974477	1.0654	828	0.974477	1.56769	9482
11	1.02777	1.10792	741	1.02777	1.56769	8654
12	1.07248	1.14399	588	1.07248	1.56769	7913
13	1.11046	1.17471	484	1.11046	1.56769	7325
14	1.1431	1.2014	459	1.1431	1.56769	6841
15	1.17145	1.22451	404	1.17145	1.56769	6382
16	1.19631	1.24532	361	1.19631	1.56769	5978
17	1.21828	1.2635	308	1.21828	1.56769	5617
18	1.23778	1.27983	282	1.23778	1.56769	5309
19	1.25527	1.29463	254	1.25527	1.56769	5027
20	1.27101	1.30804	215	1.27101	1.56769	4773

There are various numerical issues which need to be considered here. First of all, it may be that the number of samples is insufficient to estimate the parameters. We do not rigorously consider this possibility, but repeating the process with 1000 samples or with 8000 rather than 30000 samples gives the same numbers to almost two significant figures which suggests that they are reasonably accurate. The other possibility is that these numbers are an artifact of the small denominators $q_n(\omega) < 2000$ being considered. This problem is particularly acute for large N. Consider that when we encounter $a_n(\omega) = N$ we get that $q_{n+1}(\omega) \geq N q_n(\omega)$. This is partially reflected in the table by the fact that we have considerably fewer samples for large N than for small N. However, repeating the experiment with the conditions that $0 < q_n(\omega) < 200$, $200 < q_n(\omega) < 700$ and $700 < q_n(\omega) < 2000$ gives the same values to almost two significant figures. Again, this lends some support to these numbers being universal.

In summary, this approach seems to lend quite strong numerical support to conjecture 3.3.1. However, it does not suggest any obvious ways to go about proving it.

3.3.2 Boundary conformality

Let $f : \mathbb{D} \to \mathcal{M}$ be the Riemann map from the unit disc $\mathbb{D}$ to the Maskit slice $\mathcal{M}$. If we can show directly that for almost all points $\zeta \in \partial\mathbb{D}$ that f is not conformal (that is, equation 3 doesn't hold), then by the twist theorem (theorem 2.4.1), the map f must be twisting at almost all $\zeta \in \partial\mathbb{D}$.

Equation 3 implies that if f is conformal at $\zeta \in \partial\mathbb{D}$, $z_n \in \Delta$ and $z_n \to \zeta$ then

$$\lim_{n\to\infty} \frac{f(z_n) - f(\zeta)}{z_n - \zeta} \tag{6}$$

must exist and the terms of the sequence must be bounded away from 0 and ∞. We suggest that for almost all ζ there is a sequence z_n for which the terms of the sequence are not bounded away from 0 and ∞. We give the details of this sequence below, and assuming conjectures 3.1.1 and 3.2.3 prove that it has the required properties.

Given $\omega \in [0, 1]$, define a_n and p_n/q_n to be the sequence of partial quotients and partial approximants of ω. Now define

$$\alpha_n = \left|\omega - \frac{p_n}{q_n}\right|,$$

$$z_n = p_n/q_n + i\epsilon|\omega - p_n/q_n|$$

for some small ϵ, and

$$\sigma_n = \left|\frac{f(\omega) - f(z_n)}{\omega - z_n}\right|.$$

Note that the z_n are in a Stolz angle at ω of angle $\tan^{-1}\epsilon$. also note that $|\omega - z_n| \approx \alpha_n$. We show, assuming conjecture 3.1.1, that for almost all ω there is a subsequence n_m with the property that $\sigma_{n_m+m}/\sigma_{n_m} \leq 1/2^m$. Clearly, this contradicts the sequence (σ_n) being bounded away from 0 and ∞.

Fix N (we specify how exactly later). Write S_N^{m+1} for the sequence $(N\ N\ \cdots\ N)$ (where N is repeated $m+1$ times). From lemma 2.5.3, let $\mathcal{S} \subseteq [0,1]$ be the full measure set of numbers with continued fraction expansions in which every finite sequence of positive integers occurs infinitely often. In particular, for $\omega \in \mathcal{S}$ the sequence S_N^{m+1} occurs in the continued fraction expansion of ω. Define $n_m(\omega)$ so that $a_{n_m+1} = \cdots = a_{n_m+m+1} = N$.

Equation 5 shows that if $n = n_m + k$ with $0 \leq k < m$ then $\alpha_{n+1}/\alpha_n \leq (N+2)/N^3$.

Let $n = n_m + k$. Consider the trace of W_{p_n/q_n} at $f(\omega)$. Writing $\tau = q_n^{-2}(\omega - p_n/q_n)^{-1}$, we get that

$$\operatorname{Tr} W_{p_n/q_n} = 2 - \frac{\pi^2}{\tau^2} + O(\tau^{-3}). \tag{7}$$

Applying conjecture 3.1.1, we get that

$$|f(p_n/q_n) - f(\omega)| = \Theta\left(\frac{\pi^2}{\tau^2|\operatorname{Tr}'_n|}\right).$$

That is,

$$|f(p_n/q_n) - f(\omega)| = \Theta\left(\frac{\pi^2 q_n^4 |\omega - p_n/q_n|^2}{|\operatorname{Tr}'_n|}\right).$$

Applying the same reasoning at $f(z_n)$ instead of $f(p_n/q_n)$, and comparing we get that

$$|f(z_n) - f(\omega)| = \Theta\left(\frac{\pi^2 q_n^4 |\omega - p_n/q_n|^2}{|\operatorname{Tr}'_n|}\right).$$

From this we get that

$$\sigma_n = \Theta\left(\frac{\pi^2 q_n^4 \alpha_n}{|\operatorname{Tr}'_n|}\right).$$

Therefore,

$$\frac{\sigma_{n+1}}{\sigma_n} = \Theta\left(\frac{q_{n+1}^4}{q_n^4} \cdot \frac{\alpha_{n+1}}{\alpha_n} \cdot \frac{|\operatorname{Tr}'_n|}{|\operatorname{Tr}'_{n+1}|}\right).$$

Now $q_{n+1} \leq (N+1)q_n$, $\alpha_{n+1}/\alpha_n \leq (N+1)/N^3$ and by conjecture 3.2.1, $|\operatorname{Tr}'_n|/|\operatorname{Tr}'_{n+1}| = \Theta(1/N^3)$. It follows that $\sigma_{n+1}/\sigma_n = O(1/N)$. The constants involved are universal, so we can choose N large enough to make $\sigma_{n+1}/\sigma_n \leq 1/2$. With this choice of N, applying the same reasoning for each $n_m \leq n < n_m + m$ we get that $\sigma_{n_m+m}/\sigma_{n_m} \leq 1/2^m$.

This completes the argument for conjecture 3.3.1, but there are a few more things worth saying about this argument. First of all, although we have used a lot of implicit constants and less than symbols in showing $\sigma_{n+1}/\sigma_n = O(1/N)$, in fact if you assumed that the bounds were all reasonably tight then you would get $\sigma_{n+1}/\sigma_n \approx \pi^2/N$ for large enough N. This would suggest that you need to take $N = 10$ for the argument to work (that is, for $\sigma_{n+1}/\sigma_n \leq K < 1$ so that $\sigma_{n+m}/\sigma_n \to 0$ uniformly in m). Sure enough, running calculations on a computer show that $N = 9$ doesn't work and $N = 10$ does work, and that $\sigma_{n+1}/\sigma_n \approx \pi^2/N$ is reasonably accurate except for the first few N. For various computations of this sort, see the Mathematica notebook sigma-scale-ratios.nb.

Given any two sequences $A = (a_1 \ldots a_n)$ and $B = (b_1 \ldots b_k)$, define the sequence S_m to be A followed by m copies of B and p_m/q_m to be the corresponding fractions. Writing σ_m for the corresponding ratio, we get that σ_{m+1}/σ_m seems to depend almost entirely on the sequence B and not on A at all, and is approximately constant. In particular, for some sequences B this ratio is always greater than 1, and for some sequences it is always less than 1. The only obvious exception is the sequence $B = (1)$ where the ratio is sometimes less than and sometimes greater than 1. Since the subsequence consisting of m copies of B appears in the continued fraction expansion of almost every ω, it appears likely that the quantity $\sigma_n(\omega)$ varies wildly between 0 and ∞ for almost all ω (although this isn't certain, it could tend to either 0 or ∞ almost everywhere). pleating rays?

It is also worth saying something about the sequence $B = (1)$. It has been conjectured that the point on the boundary of the Maskit slice corresponding to the golden mean $\phi = (\sqrt{5}-1)/2$ (whose continued fraction is $[0\ 1\ 1\ 1\ \ldots]$) is the lowest point. If this conjecture were true (and extensive computation suggests it is), it would rule out the possibility of the Maskit slice spiralling at that point (because if the boundary spiralled around it, there would have to be a point lower than it). In fact, it seems plausible that the boundary is conformal at this point (and every irrational point whose continued fraction has a tail of 1s only). Indeed, if p_n/q_n are the partial approximants to ϕ then computer calculations suggest that the initial direction of the pleating ray at the $2n$th cusp is exactly vertical (and at the $2n+1$th cusp, approximately vertical). See the Mathematica notebook golden-mean-sequence.nb. This suggests that the irrational pleating ray at ϕ might extend beyond the boundary, which would show that the boundary was conformal at that point. For other continued fractions with a tail of 1s, it seems as though the initial directions of the pleating rays at the cusps corresponding to the partial approximants tend towards a fixed angle, or at least vary within a very narrow band, which suggests that the same may be true of these points too. This would be consistent with the ratio σ_{n+1}/σ_n varying around 1.

Finally, it is possible to relate the sequence B to the spiralling behaviour at a point. Whenever the sequence $(1\ N\ 1\ N\ \ldots\ 1\ N)$ of length $2m$ appears in the continued fraction expansion of ω, then at the scale of the cusps corresponding to the partial approximants in this subsequence, the boundary spirals around approximately $m/4$ times. Each appearance of $(1\ N)$ introduces a rotation of angle approximately $\pi/2$ in the initial direction of the pleating rays. The sequence $(N\ N\ \ldots\ N)$ doesn't correspond to any spiralling, because the first N introduces a rotation of $\pi/2$, the second one $-\pi/2$ and so forth. In particular, note that if the tail of the continued fraction expansion of ω consisted entirely of N's (or indeed of any large integers), the boundary would not be expected to spiral infinitely at this point, but it wouldn't be conformal either.

3.4 Hausdorff dimension

The conjecture concerning the statistical distribution of sizes of $|\operatorname{Tr}'_{p/q}(\mu_{p/q})|$ suggests an argument to show that the Hausdorff dimension of the boundary must be less than 1.25.

Let $U_{p/q}$ be the ball of radius $c_{p/q}$ about $\mu_{p/q}$, where

$$c_{p/q} := \frac{2}{|\operatorname{Tr}'_{p/q}(\mu_{p/q})|}.$$

Numerical analysis (in Mathematica notebook hausdorff.nb) suggests strongly the following conjecture.

Conjecture 3.4.1 *For all N,*

$$\bigcup_{\substack{p/q \\ q\geq N}} U_{p/q} \cap \mathcal{M} = \mathcal{M} - \{\mu_{p/q} : q < N\}.$$

That is, for any N, the class $\{U_{p/q} : q \geq N\}$ is a cover of $\mathcal{M}$ missing only finitely many points.

The cover is illustrated in figure 10. This conjecture is related to conjecture 3.1.1. If $U_{p/q}$ covers the image of the interval $I_{p/q} = (p/q - 1/q^2, p/q + 1/q^2)$, then the identity in conjecture 3.4.1 follows from the corresponding identity

Make a note about how this conjecture or something quite similar to it is quite likely to follow from the uniform local coordinates conjecture, because that conjecture if true implies something about almost all points in the boundary based on limits of what happens at cf convergents

$$\bigcup_{\substack{p/q \\ q\geq N}} I_{p/q} = [0,1] - \{p/q : q < N\}.$$

This identity follows directly from lemma 2.5.4. Conjecture 3.1.1 implies that the image of the interval $I^{\epsilon}_{p/q} = (p/q - \epsilon/q^2, p/q + \epsilon/q^2)$ has diameter of order $\epsilon/|\operatorname{Tr}'_{p/q}|$. Although it is true that the union of the intervals $I^{\epsilon}_{p/q}$ for all p/q with $q \geq N$ has full measure (by theorem 2.5.6), nonetheless this set could be compressed by the Riemann map into a set of measure 0 and so conjecture 3.1.1 is not sufficient.

Conjectures 3.2.3 and 3.4.1 together imply the following conjecture.

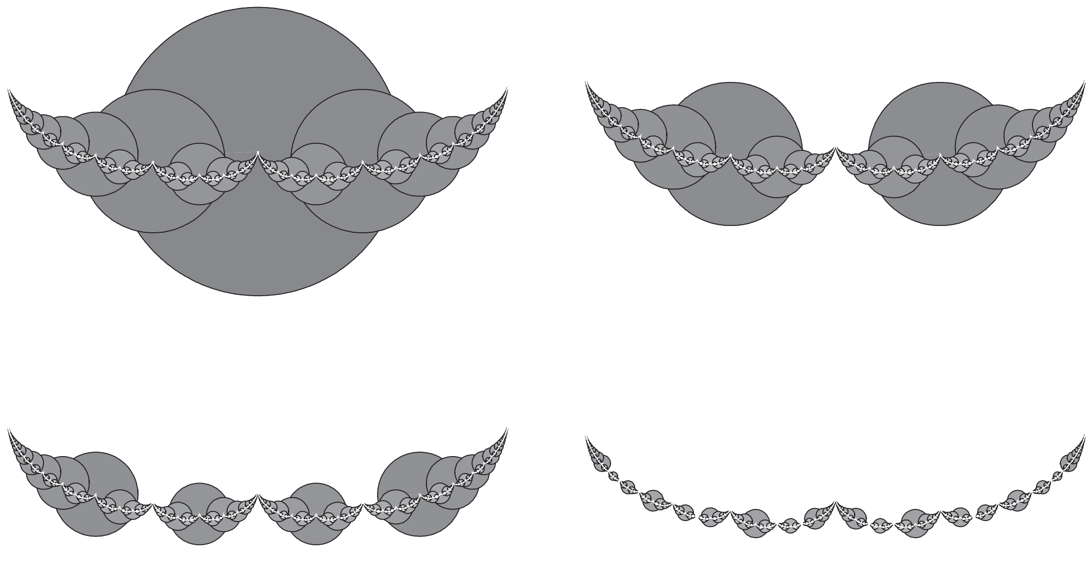

Figure 10: Covers of the Maskit slice by open balls $U_{p/q}$ with $q \geq 2$, $q \geq 3$, $q \geq 4$ and $q \geq 8$

Conjecture 3.4.2 *The Hausdorff dimension of the Maskit slice satisfies* $\dim_{\mathrm{H}}(\mathcal{M}) < 1.25$.

The Hausdorff d-dimensional measure $\Lambda_d(\mathcal{M})$ of the boundary is defined by

$$\Lambda_d(\mathcal{M}) = \lim_{\epsilon \to 0} \inf_{\substack{\text{covers } \{U_\alpha\} \\ \text{with} \\ \operatorname{diam} U_\alpha \leq \epsilon}} \sum_\alpha (\operatorname{diam} U_\alpha)^d.$$

In particular, if we define

$$\epsilon_N = \sup_{\substack{p/q \\ q \geq N}} \operatorname{diam} U_{p/q} = \sup_{\substack{p/q \\ q \geq N}} c_{p/q}$$

then $\epsilon_N \to 0$ as $N \to \infty$ (by conjecture 3.2.3), and so

$$\Lambda_d(\mathcal{M}) \leq \lim_{N \to \infty} \sum_{\substack{p/q \\ q \geq N}} c^d_{p/q}.$$

Conjecture 3.2.3 tells us that for all p/q we have that $c_{p/q} \leq 2/q^{1.6}$. In the definition of the Hausdorff measure, the constant factor 2 is irrelevant so for ease of notation we will say $c_{p/q} \leq 1/q^{1.6}$. Therefore, if $d \geq 2/1.6 + \epsilon$ then $c^d_{p/q} \leq 1/q^{2+\epsilon}$. Therefore, we have that for sufficiently large N,

$$\Lambda_d(\mathcal{M}) \;\leq\; \sum_{\substack{p/q \\ q \geq N}} c^d_{p/q} \;\leq\; \sum_{\substack{p/q \\ q \geq N}} \frac{1}{q^{2+\epsilon}} \;\leq\; \sum_{q \geq N} \frac{\phi(q)}{q^{2+\epsilon}} \;\leq\; \sum_{q \geq N} \frac{1}{q^{1+\epsilon}} \;<\; \infty.$$

Here $\phi(q)$ is the Euler totient function giving the number of integers $0 \leq p < q$ coprime to q which clearly satisfies $\phi(q) \leq q$. On average, $\phi(q) \approx 6q/\pi^2 \approx 0.6q$ so we can't improve on this. It follows that $\dim_{\mathrm{H}}(\mathcal{M}) \leq 2/1.6 = 1.25$.

Conjecture 3.2.2 suggests that on average, $c_{p/q} \leq 1/q^{1.69846}$ which suggests the bound $\dim_{\mathrm{H}}(\mathcal{M}) \leq 1.17754$ is likely to be correct.

Mention Farey series here, either cover earlier and ref or rewrite here

In fact, both these bounds seem likely to be large overestimates. Write $\mu_{q,n}$ for the cusp corresponding to the nth fraction in the qth Farey series. That is, the $\mu_{q,n}$ are the ordered $\mu_{r/s}$ for $s \leq q$. Define $S^d_q = \sum_n |\mu_{q,n} - \mu_{q,n+1}|^d$. Here we are guessing that $|\mu_{q,n} - \mu_{q,n+1}|$ is a good estimate for the diameter of the segment of $\mathcal{M}$ between $\mu_{q,n}$ and $\mu_{q,n+1}$ (which seems likely after inspecting the Maskit slice at various scales), and that $\lim_{q\to\infty} S^d_q$ is a good estimate of $\Lambda_d(\mathcal{M})$. Figure 11 shows a graph of S^d_q against q for various values of d. At $d = 1$, it seems clear that $S^d_q \to \infty$ as $q \to \infty$. At $d = 1.6$ it seems clear that $S^d_q \to 0$. The turning point between these two behaviours seems to be around $d = 1.0582$ to $d = 1.0584$. This suggests $\dim_{\mathrm{H}}(\mathcal{M}) \approx 1.058$.

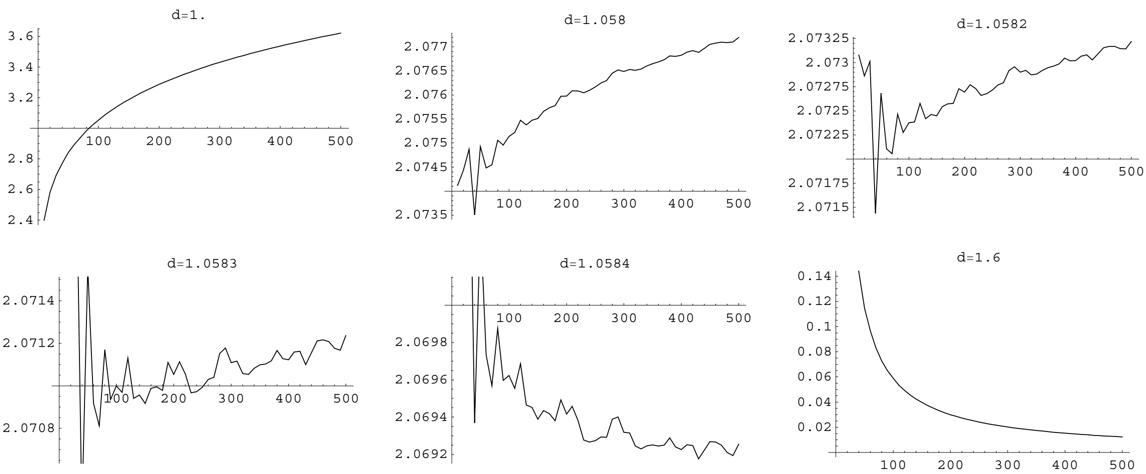

Figure 11: Graphs of estimated Hausdorff measure for various values of d

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